Depinning transition of the Mullins-Herring equation with an external driving force and quenched random disorder

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We study the depinning transition of the quenched Mullins-Herring equation by direct integration method. At critical force F_c, the average surface velocity $v(t)$ follows a power-law behavior $v(t) \sim t^{-\delta}$ as a function of time t with $\delta=0.160(5)$. The surface width has a scaling behavior with the roughness exponent $\alpha=1.50(6)$ and the growth exponent $\beta=0.841(5)$. Above the critical force, the steady state velocity v_s follows $v_s \sim (F-F_c)^{\theta}$ with $\theta=0.289(8)$. Finite size scalings of the velocity are also discussed.

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Recently, there has been considerable interest in the study of a driven interface in quenched random media. It is related through various mappings to many other physical phenomena such as the immiscible displacement of fluids in porous media [1], the domain walls in magnetic systems [2], the flux movement in a superconductor [3], and the invasion of liquid in porous media [4]. The surface dynamics is described by the coarse grained height variables $h(x,t)$ which represent the growing interface as a function of the lateral coordinate x and the time t.

An interesting quantity of the surface growth is the dynamic scaling behavior of the interface width W, which is defined as the root mean square fluctuation of the surface height. The surface width in a finite system of lateral size L, follows a scaling relation [5,6],

$$W(L,t) = \left\langle \frac{1}{L} \sum [h(x,t) - \bar{h}]^2 \right\rangle^{1/2} \sim L^{\alpha} f\left(\frac{t}{L^z}\right), \qquad (1)$$

where $\bar{h}(t)$ is the average height at time t. The scaling function $f(x)$ is x^{β} for $x \ll 1$ and $f(x)$ is constant for $x \gg 1$. The exponents β and z are connected by the relation $z\beta=\alpha$. The width increases as t^{β} initially ($t \ll L^z$) and it reaches a saturation value $W(L,t) \sim L^{\alpha}$ for $t \gg L^z$. There is a time-dependent length scale $\xi_x(t) \sim t^{1/z}$ which describes the lateral correlation of the surface height.

One of the simple equations for the interface pinning phenomena in quenched random media is the quenched Edwards-Wilkinson (QEW) equation,

$$\frac{\partial h(x,t)}{\partial t} = \nu \nabla^2 h + \eta(x,h) + F, \qquad (2)$$

where F is the external driving force and $\eta(x,h)$ is the quenched random potential satisfying the relation $\langle \eta(x,h)\eta(x',h')\rangle = 2D\delta(x-x')\delta(h-h')$. There are many studies on the QEW equation and the related models [7–13]. The $\nabla^2 h$ term in Eq. (2) describes the relaxation process of the surface height in the gravitational field where the downhill current is proportional to the height differences. Since the gravitational potential is very small compared to the chemical binding energy, one can ignore it. The motion of an atom depends on the number of connected bonds, which increases with the curvature of the site. So the surface chemical potential can be proportional to the surface curvature. If the surface current is driven by the differences in the surface chemical potential, one can replace $\nabla^2 h$ by $-\nabla^4 h$ in the QEW equation [6,14]. So it would be interesting to consider a quenched Mullins-Herring (QMH) equation [15],

$$\frac{\partial h(x,t)}{\partial t} = -K\nabla^4 h + \eta(x,h) + F. \qquad (3)$$

The first term, $-K\nabla^4 h$, describes relaxation by surface diffusion. The equation could be related to the dynamics of the liquid in porous media.

In this paper, we study the quenched Mullins-Herring equation by using a numerical direct integration method and find that $\alpha \approx 1.50$, $\beta \approx 0.841$, and $z \approx 1.78$ at the critical force. Note that the thermal MH (TMH) equation with thermal noise $\eta(x,t)$ has $\alpha=3/2$, $\beta=3/8$, and $z=4$ in one substrate dimension [16].

When an interface is driven by an external force F in disordered media, its motion shows a pinning-depinning transition. The quenched disorder generates random pinning forces effectively. If the driving force F is sufficiently weak compared to the random pinning force, the interface is pinned by the disorder. If F is strong enough, the interface moves indefinitely with the growth velocity v, $v(t) = d\bar{h}(t)/dt$. At critical force F_c, we expect that $v(t)$ decreases following

$$v(t) \sim t^{-\delta}, \qquad (4)$$

where t is the time.

In the vicinity of the depinning transition, the average velocity at the stationary state follows

$$v_s(F) \sim (F - F_c)^{\theta} \qquad (5)$$

for $F > F_c$, where θ is the velocity exponent.

We can define various correlation lengths for a growing surface. Near F_c, there is a lateral correlation length $\xi_x(F)$, which diverges as F approaches the critical force F_c,

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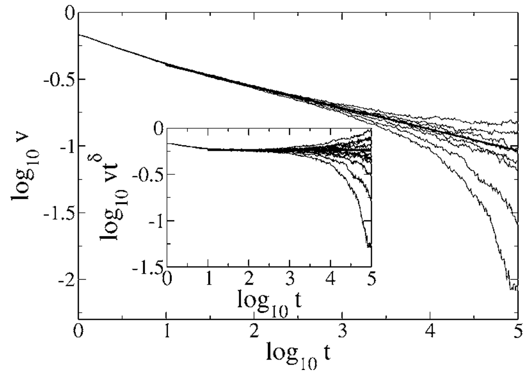

FIG. 1. Log-log plot of $v(t)$ for $L=16384$ and $F=0.985$, 0.986, 0.9872, 0.9876, 0.988, 0.9884, 0.9888, and 0.99 from the top to the bottom. The straight line is for $F=0.988$. Log-log plot of vt^{δ} against time t with $\delta=0.160$ is shown in the inset.

$$\xi_x(F) \sim |F - F_c|^{-\nu_x}. \tag{6}$$

One can consider a correlation length in the height direction $\xi_h(F)$, which follows

$$\xi_h(F) \sim |F - F_c|^{-\nu_h}. \tag{7}$$

Since it should be proportional to the surface width, $\xi_h \sim W \sim (\xi_x)^{\alpha}$, there is a relation $\alpha=\nu_h/\nu_x$. The correlation time $\tau(F)$ can be defined as

$$\tau(F) \sim |F - F_c|^{-\nu_t}, \tag{8}$$

which describes the correlation in the time direction. From the above relations, one can obtain $z=\nu_t/\nu_x$. It should be the same as the dynamic exponent described in the surface width of Eq. (1) because there is only one dynamic exponent in the system.

We study the QMH equation (3) by a direct numerical integration method,

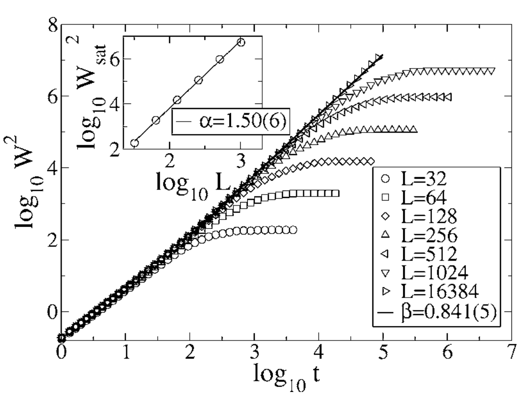

FIG. 2. Log-log plot of the surface width against time at $F_c=0.988$. The saturated width W_{sat} against L are shown in the inset.

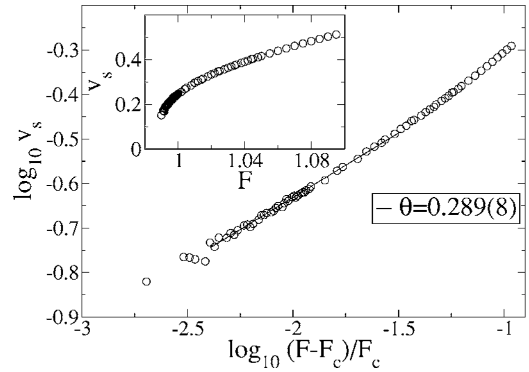

FIG. 3. Log-log plot of the steady state velocity v_s as a function of $F-F_c$ for $L=4096$. v_s against F in linear scales is shown in the inset.

$$h(x,t+\Delta t) = h(x,t) + \Delta t\{F + \eta(x,\tilde{h}) - K[h(x-2,t) - 4h(x-1,t) + 6h(x,t) - 4h(x+1,t) + h(x+2,t)]\}. \tag{9}$$

The noise $\eta(x,\tilde{h})$ is uniformly distributed in $[-\sqrt{3},\sqrt{3}]$ where $\tilde{h}$ is the integer part of $h(x,t)$. For simplicity, $K=1$ is chosen, and the time steps used are $\Delta t=0.01$ in most cases. The change in the time steps is not expected to change the exponents. We verify that smaller time step ($\Delta t=0.001$) did not change the numerical results. The external force F is a control parameter of the growth dynamics. The integration processes are performed in one substrate dimension with a periodic boundary condition on the system of size L. The height of each site is updated simultaneously following Eq. (9).

We measure the growth velocity as a function of time for various values of the external driving force F. We find $F_c=0.988$ where $v(t)$ shows a power-law behavior following Eq. (4). As shown in Fig. 1, in the depinned region $F>F_c$, v decreases with time at the beginning, and then it becomes a constant at the later time. While in the pinned region, $F<F_c$, v decays faster than a power-law behavior and becomes zero eventually. At the critical force, $v(t)$ follows a power law $t^{-\delta}$ with

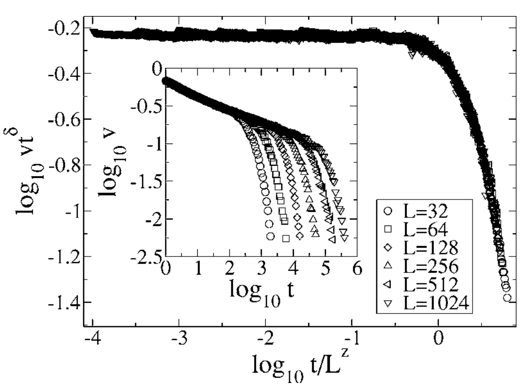

FIG. 4. The scaling plot according to Eq. (14) with $\delta=0.160$ and $z=1.78$ at F_c. Log-log plot of v against t is shown in the inset.

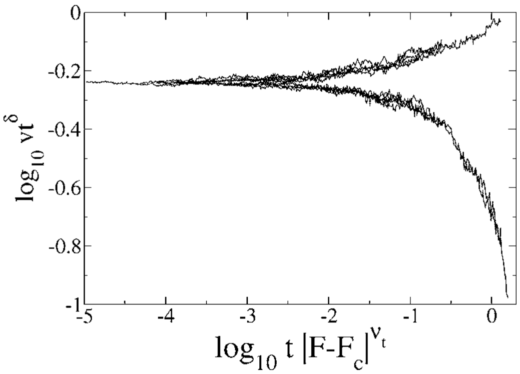

FIG. 5. The scaling plot of the data in Fig. 1 following Eq. (15) with δ=0.160 and ν_t=1.81 (L=16384).

$$\delta = 0.160(5). \tag{10}$$

In the inset of Fig. 1, vt^{δ} against t in log-log plot is given. With δ=0.160, we get a nice straight line at F_c=0.988. For $F > F_c$, it curves upward as a function of time while it goes downward for $F < F_c$. So there is a phase transition between the pinned phase and the depinned phase as varying the external force.

At the transition point, we monitor surface width as a function of time for various system sizes. As shown in Fig. 2, it increases as t^{β} for early time and eventually saturates when the parallel correlation ξ_x is of the order of the lateral system size L. For the roughness exponent α describing the saturation of the interface fluctuation, we use the relation $W_{sat} \sim L^{\alpha}$ in the steady state region $t \gg L^z$ as shown in the inset of Fig. 2 and obtain

$$\alpha = 1.50(6), \quad \beta = 0.841(5), \quad \text{and } z \approx 1.78. \tag{11}$$

Near F_c, the surface is affected by the quenched noise. At or below F_c, some part of the surface is pinned. Therefore, one can consider that $\bar{h}(t)$ is approximately proportional to $W(t)$. Since $\bar{h}(t) \sim W(t) \sim t^{\beta}$ for early time, $v(t) = d\bar{h}(t)/dt \sim t^{\beta-1}$ at F_c. From Eq. (4) we get a relation, $\beta + \delta = 1$ [17]. Our exponents, β=0.841 and δ=0.160, satisfy the relation very well.

In the depinned regime the surface grows, and the growth velocity becomes constant. Above F_c, we measure the steady state velocity as a function of F for a system size L=4096 as shown in Fig. 3. From the relation Eq. (5), we obtain

$$\theta = 0.289(8). \tag{12}$$

θ is expected to be $\nu_t - \nu_h$ since $v_s \sim \xi_h/\tau \sim (F - F_c)^{\nu_t - \nu_h}$. Also, δ and θ have the relation $\delta = \theta/\nu_t$ [18].

We obtain the exponents α, β, and θ from the numerical simulation data. The other critical exponents are estimated by several scaling relations $z = \alpha/\beta$, $\nu_t = \theta/\delta$, $\nu_x = \nu_t/z$, and $\nu_h = \alpha\nu_x$. Therefore, we get,

$$z \approx 1.78, \quad \nu_t \approx 1.81, \quad \nu_x \approx 1.01, \quad \text{and } \nu_h \approx 1.52. \tag{13}$$

These exponents satisfy the relations, $\theta = \nu_t - \nu_h \approx 0.29$ and $\beta = \alpha/z = \nu_h/\nu_t \approx 0.84$ within the error bar.

The validity of the scaling relations is further studied by the finite size scaling of the velocity $v(t)$. At the critical force, $v(L,t)$ follows

$$v(L,t) \sim t^{-\delta}\Phi_t(t/L^z). \tag{14}$$

We plot $v(L,t)t^{-\delta}$ against t/L^z with δ=0.16 and z=1.78 as shown in Fig. 4 and get a nice data collapse suggesting that our values of the critical exponents are consistent with each other.

Near the critical force, $v(F,t)$ for a sufficiently large system has a scaling formula

$$v(F,t) \sim t^{-\delta}\Phi_f(t|F - F_c|^{\nu_t}). \tag{15}$$

The plot of $v(F,t)t^{\delta}$ against $t|F - F_c|^{\nu_t}$ is given in Fig. 5. Our scaled data are collapsed onto two different curves with δ=0.16 and ν_t=1.81 for L=16384. The collapsed data for $F > F_c$ are curved upward, and the data for $F < F_c$ are curved downward.

We study the depinning transition of the MH equation in random media. Since the growth velocity follows a power-law behavior as a function of time at the critical point, we can determine the critical force accurately. Using the numerical integration of the QMH equation, α=1.50(6), β=0.841(5), δ=0.160(5), and θ=0.289(8) are obtained at $F_c \approx 0.988$. The other exponents $z \approx 1.78$, $\nu_t \approx 1.81$, $\nu_x \approx 1.01$, and $\nu_h \approx 1.52$ are estimated through the relations, $z = \alpha/\beta$, $z = \nu_t/\nu_x$, $\delta = \theta/\nu_t$, and $\alpha = \nu_h/\nu_x$. It is interesting that the value of the dynamic exponent $z \approx 1.78$ is quite small compared to z=4 of the thermal MH equation. The measured values of the exponents in our work satisfy the scaling relations $\beta + \delta = 1$ and $\theta = \nu_t - \nu_h$ very well. Therefore, the surface growth in the quenched noise can be classified by only three independent exponents α, β, and θ. All the other critical exponents can be reduced from them. Experiments and analytic calculation of the exponents for the QMH equation are required.

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